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New solutions for effective elastic moduli of microcracked solids

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Abstract

A new energy balance equation is proposed to evaluate the effective moduli of solids with randomly dispersed cracks. In this study, it is assumed that the potential energy released by embedding a circular or spherical RVE with microcracks into an infinite matrix is equal to that induced by introducing its effective medium into the identical infinite matrix. New non-interacting and self-consistent solutions for the effective moduli of linear elastic isotropic solids with randomly oriented microcracks were evaluated on the basis of the present energy balance equation. The linearization of the two present solutions leads to the dilute solution in the limit of small crack density. Comparison studies with various existing solutions are also presented in detail.

Keywords: New energy balance equation; Effective moduli of microcracked solids; Non-interacting solution; Self-consistent solution

1. Introduction

Many approximate schemes for determining effective moduli of cracked solids have been proposed in literature. Bristow (1960) was the first to obtain a non-interacting solution for microcracked solids. The self-consistent method for inhomogeneous materials (Hershey and Dahlgren, 1954; Kroner, 1958; Hill, 1965; Budiansky, 1965) was used to calculate the effective moduli of solids with random cracks by Budiansky and O'Connell (1976) and it was further explored by many researchers (Hoenig, 1979; Horii and Nemat-Nasser, 1983; Horii and Nemat-Nasser, 1990; Gottesman et al., 1980; Laws et al., 1983; Laws and Brockenbrough, 1987; Laws and Dvorak, 1987; Sumarac and Krajcinovic, 1987; Sumarac and Krajcinovic, 1989; Krajcinovic and Sumarac, 1989; Ju, 1991; Ju and Lee, 1991; Lee and Ju, 1991). Kachanov (1987) developed a numerical method to compute the effective moduli of cracked solids.

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Kachanov (1992, 1994) also evaluated various existing approximate models based on his numerical results. Huang et al. (1996) presented the numerical solutions. The generalized self-consistent method for composite materials (Christensen and Lo, 1979) was adopted for cracked solids by Aboudi and Benveniste (1987) and Huang et al. (1994). Aboudi and Benveniste (1987) embedded a crack in a circular matrix that, in turn, was put into an effective medium and Huang et al. (1994) explored an elliptical (2-D) or an ellipsoidal (3-D) matrix. Hashin (1988) also evaluated the effective moduli of microcracked solids using the differential method (Roscoe, 1952, 1973; McLaughlin, 1977; Norris, 1985; Laws and Dvorak, 1987). The Mori–Tanaka method (Mori and Tanaka, 1973) was utilized for microcracked solids by Zhao et al. (1989) and Benveniste (1986).

In this study, a new energy balance equation is proposed. In addition, new non-interacting and self-consistent solutions for the effective elastic moduli of solids with randomly oriented microcracks are presented.

2. A new energy balance equation for microcracked solids

As shown in Fig. 1(a,b), the circular (2-D) or spherical (3-D) RVE with microcracks in the infinite matrix and its effective medium in the infinite matrix are considered. In the present study, it is assumed that the potential energy released by embedding the circular or spherical RVE with microcracks into the infinite matrix is equal to that induced by introducing its effective medium into the identical infinite matrix:

$$\Delta f_{\text{effective}} = \Delta f_{\text{micro}}, \tag{1}$$

where $\Delta f_{\text{effective}}$ and Δf_{micro} are the potential energies released by the effective medium and the microcracks embedded in the infinite matrix, respectively. $\Delta f_{\text{effective}}$ can be obtained by Eshelby's method as (Eshelby, 1957)

$$\Delta f_{\text{effective}} = \frac{1}{2} A \boldsymbol{\sigma}^0 : \left[\mathbf{C}_0 : (\mathbf{C} - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{C}_0 : S_0 \right]^{-1} : \boldsymbol{\sigma}^0, \tag{2}$$

where $\boldsymbol{\sigma}^0$ are the far-field stresses, $\mathbf{C}$ and $\mathbf{C}_0$ are the effective elastic stiffness tensor of the microcracked solid and the elastic stiffness tensor of the matrix, respectively; A denotes the area of the circular RVE for 2-D problems, which also denotes the corresponding sub-region in terms of the context; $\mathbf{S}_0$ is Eshelby's tensor associated with the Poisson's ratios of the matrix and the circular cylinder region.

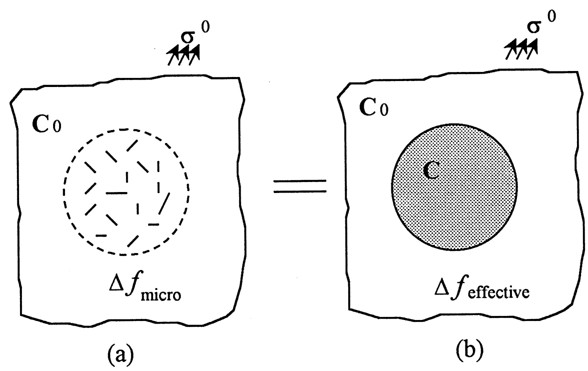

Fig. 1. A new energy balance for the analysis of effective moduli of microcracked solids.

Subsequently, Eqs. (2) and (1) yield a new energy balance equation:

$$\frac{1}{2}A\boldsymbol{\sigma}^0:\left[\mathbf{C}_0:(\mathbf{C}-\mathbf{C}_0)^{-1}:\mathbf{C}_0+\mathbf{C}_0:\mathbf{S}_0\right]^{-1}:\boldsymbol{\sigma}^0=\Delta f_{\text{micro}}. \tag{3}$$

The solutions for effective shear and bulk moduli of elastic isotropic solids with randomly oriented cracks can be evaluated using Eq. (3).

Consider the microcracked solid under two remote loading conditions such as plane hydrostatic stress $\boldsymbol{\sigma}_K^0$ with $\sigma_{K\alpha\beta}^0=\sigma_K^0\delta_{\alpha\beta}$ ($\alpha=1$, 2; $\beta=1$, 2) and in-plane pure shear $\boldsymbol{\sigma}_G^0$ with $\sigma_{G11}^0=-\sigma_{G22}^0=\sigma_G^0$ and other $\sigma_{G\alpha\beta}^0=0$. The effective medium is taken as isotropic. Then two independent equations for the effective plane bulk modulus K and the effective shear modulus G of the microcracked solid can be obtained from Eq. (3) as

$$\frac{1}{2K_0}\frac{K-K_0}{K_0+\xi(K-K_0)}=\frac{1}{\left(\sigma_K^0\right)^2}\frac{1}{A}\Delta f_{\text{micro}} \tag{4}$$

and

$$\frac{1}{2G_0}\frac{G-G_0}{G_0+\eta(G-G_0)}=\frac{1}{\left(\sigma_G^0\right)^2}\frac{1}{A}\Delta f_{\text{micro}}, \tag{5}$$

where $K_0=E_0/[2(1+\nu_0)(1-2\nu_0)]$ and $G_0=E_0/[2(1+\nu_0)]$ denote the plane strain bulk and shear moduli of the matrix, respectively; E_0 and ν_0 are the Young's modulus and Poisson's ratio of the matrix; $\xi=1/[2(1-\nu_0)]$, $\eta=(3-4\nu_0)/[4(1-\nu_0)]$ for 2-D plane strain problems. Similarly, the solutions for 3-D problems can be evaluated by replacing the circular RVE with the spherical RVE. Eshelby's tensor $\mathbf{S}_0$ will then be associated with Poisson's ratios of the matrix and the spherical region, and $\xi=(1+\nu_0)/[3(1-\nu_0)]$, $\eta=(8-10\nu_0)/[15(1-\nu_0)]$.

Note that Δf_{micro} in the present energy balance equation (Eqs. (3)–(5)) is the potential energy release by embedding the circular or spherical RVE containing microcracks into the infinite matrix.

3. Non-interacting approximation

It is difficult to solve microcrack-interaction problems exactly. However, Δf_{micro} can be easily evaluated based on the approximation of non-interacting cracks. The potential energy release Δf_{micro} induced by embedding cracks into the circular sub-region of the infinite matrix is approximated as the sum of the potential energy releases induced by each single crack. Consequently, Δf_{micro} for 2-D and 3-D isotropic problems can be written as (Kachanov, 1992, 1994)

$$\Delta f_{\text{micro}}=-A\left(\frac{\pi}{2E_0'}\right)\rho\sigma_{ij}^0\sigma_{ij}^0 \tag{6}$$

and

$$\Delta f_{\text{micro}}=-V\frac{8\left(1-\nu_0^2\right)}{9(1-\nu_0/2)E_0}\rho\left[\left(1-\frac{\nu_0}{5}\right)\sigma_{ij}^0\sigma_{ij}^0-\frac{\nu_0}{10}\left(\sigma_{kk}^0\right)^2\right], \tag{7}$$

where V is the volume of the spherical RVE and $E_0'=E_0/(1-\nu_0^2)$.

The conventional non-interacting solution for effective moduli of microcracked solids is (Bristow, 1960)

$$\frac{K}{K_0} = 1 \Big/ \left(1 + \frac{1 - \nu_0}{1 - 2\nu_0}\pi\rho\right), \tag{8}$$

$$\frac{G}{G_0} = 1/[1 + (1 - \nu_0)\pi\rho] \tag{9}$$

and

$$\frac{K}{K_0} = 1 \Big/ \left[1 + \frac{16}{9}\frac{1 - \nu_0^2}{1 - 2\nu_0}\rho\right], \tag{10}$$

$$\frac{G}{G_0} = 1 \Big/ \left[1 + \frac{16}{9}\frac{1 - \nu_0}{1 - \nu_0/2}\left(1 - \frac{\nu_0}{5}\right)\rho\right]. \tag{11}$$

Note that the approximation of small crack density is not equivalent to the approximation of the non-interacting cracks (Kachanov, 1992, 1993).

Substituting Eqs. (6) and (7) into the present energy balance equations (Eqs. (4) and (5)) yields a new non-interacting solution for 2-D and 3-D problems

$$\frac{K}{K_0} = 1 \Big/ \left[1 + \frac{1 - \nu_0}{1 - 2\nu_0}\pi\rho \Big/ \left(1 - \frac{\pi}{2}\rho\right)\right], \tag{12}$$

$$\frac{G}{G_0} = 1 \Big/ \left[1 + (1 - \nu_0)\pi\rho \Big/ \left(1 - \frac{\pi}{4}\rho\right)\right] \tag{13}$$

and

$$\frac{K}{K_0} = \frac{1}{1 + \frac{16}{9}\frac{1 - \nu_0^2}{1 - 2\nu_0}\rho \Big/ \left[1 - \frac{32}{27}(1 + \nu_0)\rho\right]}, \tag{14}$$

$$\frac{G}{G_0} = \frac{1}{1 + \frac{16}{9}\frac{1 - \nu_0}{1 - \nu_0/2}\left(1 - \frac{\nu_0}{5}\right)\rho \Big/ \left[1 - \frac{16}{135}\frac{7 - 5\nu_0}{1 - \nu_0/2}(1 - \nu_0/5)\rho\right]}. \tag{15}$$

Note that the linearization of the present non-interacting solutions of Eqs. (12)–(15) leads to the dilute solution.

4. Self-consistent approximation

The non-interacting approximation completely neglects the interactions among cracks while the self-consistent method (SCM) takes account for the interactions. The conventional self-consistent solution for 2-D and 3-D problems are (Budiansky and O'Connel, 1976; Huang et al., 1996)

$$\frac{K}{K_0} = (1 - 2\nu_0)(1 - \pi\rho)/(1 - 2\nu_0 + \nu_0\pi\rho), \tag{16}$$

$$\frac{G}{G_0} = (1 - \pi\rho)/(1 + \nu_0\pi\rho) \tag{17}$$

and

$$\frac{K}{K_0} = 1 - \frac{16}{9}\frac{1 - \nu^2}{1 - 2\nu}\rho, \tag{18}$$

$$\frac{G}{G_0} = 1 - \frac{16}{9}\frac{1 - \nu}{1 - \nu/2}\left(1 - \frac{\nu}{5}\right)\rho, \tag{19}$$

with

$$\rho = \frac{45}{16}\frac{(\nu_0 - \nu)(2 - \nu)}{(1 - \nu^2)[10\nu_0 - \nu(1 + 3\nu_0)]}. \tag{20}$$

The basic idea of the self-consistent method (Budiansky and O'Connel, 1976) is utilized to calculate the $\Delta f_{\mathrm{micro}}$ in the right-hand-side of the present energy balance equations (Eqs. (4) and (5)). Budiansky and O'Connel (1976) assumed that each crack was embedded into the infinite effective medium to consider the interactions among cracks. However, as shown in Fig. 2(b), the present self-consistent method takes the circular or spherical RVE as the unknown effective media, where each crack is embedded into it. Since the cracks are very small compared with the size of the RVE, it is assumed that they are surrounded by the infinite effective media as shown in Fig. 2(c,d). Consequently, $\boldsymbol{\sigma}^0_{\mathrm{RVE}}$ on the

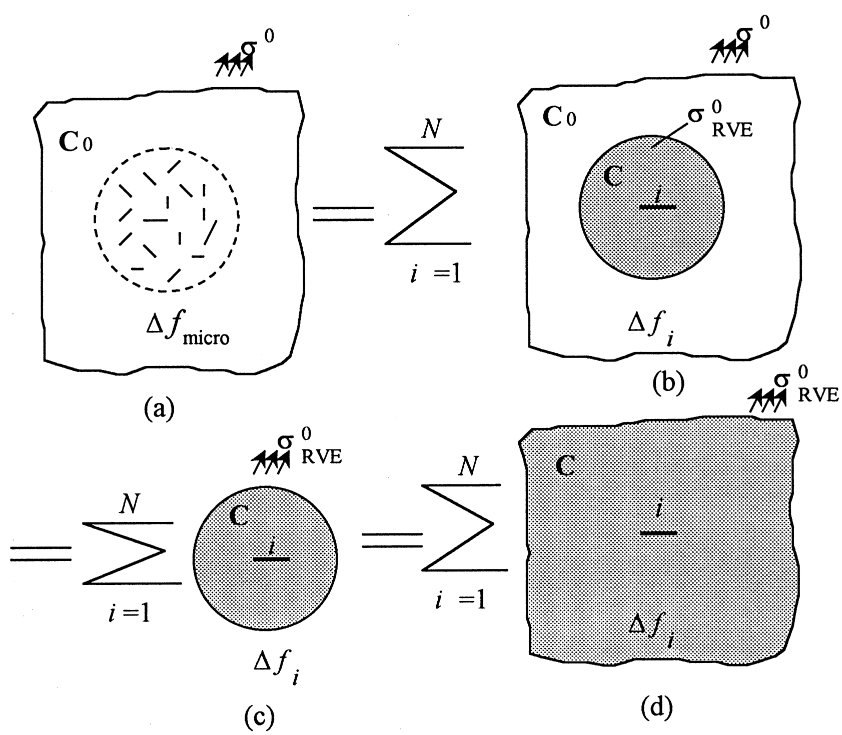

Fig. 2. The present self-consistent method for effective moduli of microcracked solids

interface between the RVE and the infinite matrix become the boundary stresses in Fig. 2(c,d), and can be calculated by Eshelby's method (Eshelby, 1957; Mura, 1982)

$$\boldsymbol{\sigma}^0_{\mathrm{RVE}} = \mathbf{C}{:}\left[\mathbf{I} + \mathbf{S}_0{:}\mathbf{C}_0^{-1}{:}(\mathbf{C} - \mathbf{C}_0)\right]^{-1}{:}\mathbf{C}_0^{-1}{:}\boldsymbol{\sigma}^0. \tag{21}$$

When the plane hydrostatic (2-D) or hydrostatic (3-D) stresses $\sigma^0_{ij} = \sigma^0\delta_{ij}$ or in-plane pure shear stresses $\sigma^0_{11} = -\sigma^0_{22} = \tau^0$ and other $\sigma^0_{ij} = 0$ are applied, the boundary stresses become $\sigma^0_{\mathrm{RVE}_{ij}} = \sigma^0_{\mathrm{RVE}}\delta_{ij}$ or $\sigma^0_{\mathrm{RVE}_{11}} = -\sigma^0_{\mathrm{RVE}_{22}} = \tau^0_{\mathrm{RVE}}$ and other $\sigma^0_{\mathrm{RVE}_{ij}} = 0$, respectively. In this case, σ^0_{RVE} and τ^0_{RVE} can be explicitly obtained from Eq. (21) as

$$\sigma^0_{\mathrm{RVE}} = \frac{K}{K_0 + \xi(K - K_0)}\sigma^0 \tag{22}$$

and

$$\tau^0_{\mathrm{RVE}} = \frac{G}{G_0 + \eta(G - G_0)}\tau^0. \tag{23}$$

By replacing E_0 and ν_0 with those of the effective medium and $\boldsymbol{\sigma}^0$ with $\boldsymbol{\sigma}^0_{\mathrm{RVE}}$ in the right-hand-sides of Eqs. (6) and (7), $\Delta f_{\mathrm{micro}}$, given by the present self-consistent method, can be obtained for 2-D as

$$\Delta f_{\mathrm{micro}} = -A(\pi/2E')\rho\sigma^0_{\mathrm{RVE}_{ij}}\sigma^0_{\mathrm{RVE}_{ij}} \tag{24}$$

and for 3-D as

$$\Delta f_{\mathrm{micro}} = -V\frac{8(1-\nu^2)}{9(1-\nu/2)E}\rho\left[\left(1 - \frac{\nu}{5}\right)\sigma^0_{\mathrm{RVE}_{ij}}\sigma^0_{\mathrm{RVE}_{ij}} - \frac{\nu_0}{10}\left(\sigma^0_{\mathrm{RVE}_{kk}}\right)^2\right]. \tag{25}$$

Then by substituting Eqs. (24) and (25) with Eqs. (22) and (23) into Eqs. (4) and (5), the bulk and shear moduli of 2-D and 3-D microcracked solids can be written as

$$\bar{K} = 1 - \frac{\pi\rho}{2}\left[1 + \frac{1}{1-2\nu_0}\frac{\bar{K}}{\bar{G}}\right]\frac{\bar{K}}{1 + 1/[2(1-\nu_0)](\bar{K}-1)}, \tag{26}$$

$$\bar{G} = 1 - \frac{\pi\rho}{2}\left[1 + (1-2\nu_0)\frac{\bar{G}}{\bar{K}}\right]\frac{\bar{G}}{1 + (3-4\nu_0)/[4(1-\nu_0)](\bar{G}-1)}, \tag{27}$$

and

$$\bar{K} = 1 - \frac{16}{9}\rho\frac{1-\nu^2}{1-2\nu}\frac{\bar{K}}{1 + (1+\nu_0)/[3(1-\nu_0)](\bar{K}-1)}, \tag{28}$$

$$\bar{G} = 1 - \frac{16}{9}\rho\frac{1-\nu}{1-\nu/2}(1-\nu/5)\frac{\bar{G}}{1 + (8-10\nu_0)/[15(1-\nu_0)](\bar{G}-1)}, \tag{29}$$

$$\nu = \left(\bar{K} - \frac{1-2\nu_0}{1+\nu_0}\bar{G}\right)/(2\bar{K} + \frac{1-2\nu_0}{1+\nu_0}\bar{G}), \tag{30}$$

where $\bar{K} = \frac{K}{K_0}$ and $\bar{G} = \frac{G}{G_0}$ are the normalized effective plane strain bulk and shear moduli for 2-D

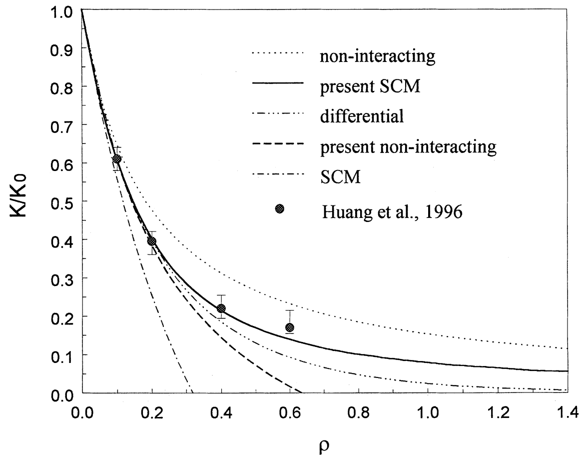

Fig. 3. Effective plane strain bulk moduli for the 2-D case

problems or the normalized effective bulk and shear moduli for 3-D problems; and ν_0 and ν are the Poisson's ratios of the matrix and the effective medium. The equations can be solved numerically and the present self-consistent solutions are reduced to the dilute solution after the linearization in the limit of small crack density.

5. Results and discussion

The new non-interacting and self-consistent solutions for effective moduli of 2-D and 3-D microcracked solids are evaluated and plotted in Figs. 3–6. The present solutions are also compared with various existing results.

The present non-interacting solutions are quite different from the conventional non-interacting solutions. However, as shown in Figs. 3 and 4, the present non-interacting solutions for the bulk and

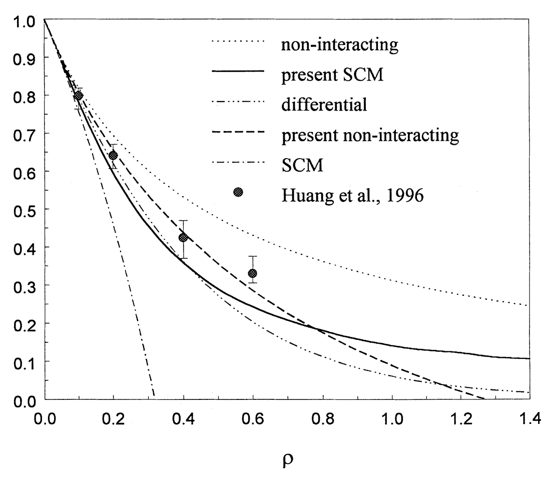

Fig. 4. Effective shear moduli for the 2-D case

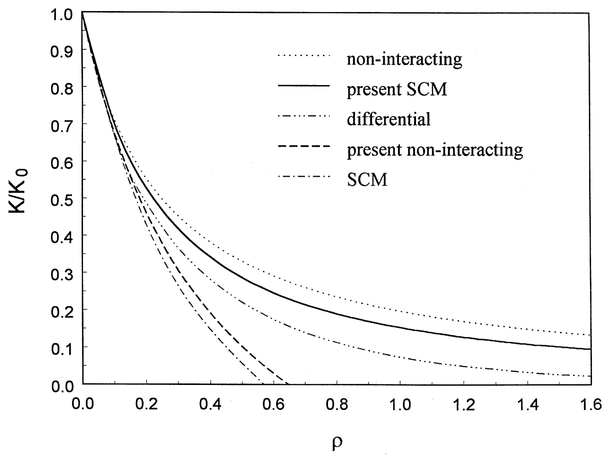

Fig. 5. Effective bulk moduli for the 3-D case

shear moduli agree well with the numerical results by Huang et al. (1996) up to the crack density $\rho = 0.2$ and 0.6, respectively. Physically, if $\rho \rightarrow \infty$, the effective moduli of the RVE should be zero. For high crack density, it can be seen that the bulk and shear moduli become zero at the cut-off points $\rho = 2/\pi$ and $\rho = 4/\pi$ for 2-D case and at $\rho \approx 0.65$ and $\rho \approx 1.39$ with $\nu_0 = 0.3$ for the 3-D case, respectively. In the present non-interacting solution, the neglected inter-crack interactions yield the cut-off points.

The drawback of the present non-interacting solution can be overcome by considering the inter-crack interactions. The present self-consistent solution accounts for the inter-crack interactions by considering the RVE with each crack as the unknown effective medium. As depicted in Figs. 5 and 6, the present self-consistent solution for plane strain bulk moduli shows an excellent agreement with the numerical results by Huang et al. (1996). However, the discrepancy is observed between the present self-consistent solution and their numerical results for shear moduli. The results show that the present non-interacting solution is generally lower than the present self-consistent solution. However, for shear moduli, both solutions cross each other.

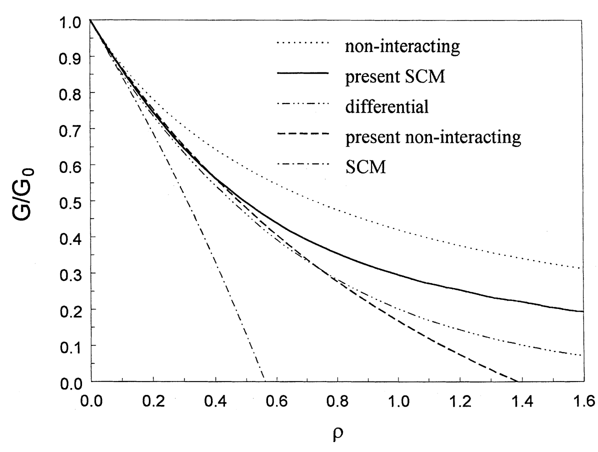

Fig. 6. Effective shear moduli for the 3-D case

6. Conclusions

Based on the present energy balance equation proposed in the present study, two new approximate models such as non-interacting and self-consistent solutions are presented. The present self-consistent solution accounts for the inter-crack interactions by considering the RVE around each crack as the unknown effective medium. The present non-interacting solution shows the cut-off points for very high crack densities.

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